Monomorphisms between Cayley-Dickson Algebras

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Abstract: In this paper we study the algebra monomorphisms from $\mathbb{A}_m = \mathbb{R}^{2^m}$ into $\mathbb{A}_n = \mathbb{R}^{2^n}$ for $1 \leq m \leq n$, where $\mathbb{A}_n$ are the Cayley-Dickson algebras. For $n \geq 4$, we show that there are many types of monomorphisms and we describe them in terms of the zero divisors in $\mathbb{A}_n$.

Key words and phrases: Cayley-Dickson, Quaternions, Octonions, non-associative, alternative, flexible and G_2.
AMS Subject classification: 17A99

Introduction: The Cayley-Dickson algebras $\mathbb{A}_n$ over the real numbers is an algebra structure on $\mathbb{R}^{2^n} = \mathbb{A}_n$ for $n \geq 0$.

By definition the Cayley-Dickson algebras (C-D algebras) are given by doubling process of Dickson [D].

For (a, b) and (x, y) in $\mathbb{A}_n \times \mathbb{A}_n$, define the product in $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$ as follows:

$$(a, b) \cdot (x, y) = (ax - \overline{y}b, ya + b\overline{x}).$$

So if $\mathbb{A}_0 = \mathbb{R}$ and $\overline{x} = x$ for all x in $\mathbb{R}$ then $\mathbb{A}_1 = \mathbb{C}$ the complex numbers $\mathbb{A}_2 = \mathbb{H}$ the quaternion numbers and $\mathbb{A}_3 = \o$ the octonion numbers.

As is well known $\mathbb{A}_n$ is commutative for $n \leq 1$; associative for $n \leq 2$ and alternative for $n \leq 3$, also $\mathbb{A}_n$ is normed for $n \leq 3$.

For $n \geq 4 \quad \mathbb{A}_n$ is flexible and has zero divisors [Mo$_1$].

Let $Aut(\mathbb{A}_n)$ be the automorphism group of the algebra $\mathbb{A}_n$.

As is well known

$Aut(\mathbb{A}_1) = \mathbb{Z}/2 = \{$ Identity, Conjugation$\}$.

$Aut(\mathbb{A}_2) = SO(3)$ the rotation group in $\mathbb{R}^3$

$Aut(\mathbb{A}_3) = G_2$ the exceptional Lie group.

(See [H-Y] and [Wh]).

For $n \geq 4$, Eakin-Sathaye showed that

$$Aut(\mathbb{A}_n) = Aut(\mathbb{A}_{n-1}) \times \sum\nolimits_3 .$$

Where $\sum_3$ is the symmetric group of order 6.

(See [E-K] and [Mo$_2$]).

In this paper we will extend the above results in the following sense:

Suppose that $1 \leq m \leq n$.

By definition an algebra monomorphism $\varphi : \mathbb{A}_m \to \mathbb{A}_n$ is a linear monomorphism such that (i) $\varphi(e_0) = e_0$ and (ii) $\varphi(xy) = \varphi(x)\varphi(y)$ for all x and y in $\mathbb{A}_m$ where $e_0 = (1, 0, \ldots, 0)$ is the unit element in $\mathbb{A}_m$ and $\mathbb{A}_n$ respectively.

We will describe the set

$$\mathcal{M}(\mathbb{A}_m, \mathbb{A}_n) = \{\varphi : \mathbb{A}_m \to \mathbb{A}_n | \varphi \quad \text{algebra monomorphism}\}.$$

We will see that this set is more complicated to describe for $n \geq 4$.

For $n \leq 3$ we will recover known results about the relationship between $Aut(\mathbb{A}_n)$ and the Stiefel manifolds $V_{2^n-1,2}$.

For $n \geq 3$, recall that $\{e_0, e_1, \ldots, e_{2^n-1}\}$ denotes the canonical basis in $\mathbb{A}_n$ and that the doubling process is given by

$$\mathbb{A}_{n+1} = \mathbb{A}_n \oplus \mathbb{A}_n \widetilde{e}_0$$

where $\tilde{e}_0 := e_{2^n}$ (half of the way basic in $\mathbb{A}_{n+1}$).

For $\varphi \in \mathcal{M}(\mathbb{A}_m; \mathbb{A}_{n+1})$ for $n \geq 3$,

φ is of type I if $e_{2^n} = \tilde{e}_0 \in (Im\varphi) \subset \mathbb{A}_{n+1}$ and φ is of type II if also $e_{2^n-1} := \varepsilon \in (Im\varphi) \subset \mathbb{A}_{n+1}$.

The main result of this paper is Theorem 2.5: the set of type II monomorphisms from $\mathbb{A}_3$ to $\mathbb{A}_{n+1}$ can be described by the set of zero divisors in $\mathbb{A}_{n+1}$ for $n \geq 4$.

§1. Pure and doubly pure elements in $\mathbb{A}_{n+1}$.

Throughout this paper we will stablish the following notational conventions:Elements in $\mathbb{A}_n$ will be denoted by Latin characters $a, b, c, \ldots, x, y, z$. Elements in $\mathbb{A}_{n+1}$ will be denoted by Greek characters $\alpha, \beta, \gamma, \ldots$ For example,

$$\alpha = (a, b) \in \mathbb{A}_n \times \mathbb{A}_n.$$

When we need to represent elements in $\mathbb{A}_n$ as elements in $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ we use subscripts, for instance, $a = (a_1, a_2), \quad b = (b_1, b_2)$, and so on, with a_1, a_2, b_1, b_2 in $\mathbb{A}_{n-1}$.

Now $\{e_0, e_1, \ldots, e_{2^n-1}\}$ denotes the canonical basis in $\mathbb{A}_n$. Then by the doubling process

$$\{(e_0, 0), (e_1, 0), \ldots, (e_{2^n-1}, 0), (0, e_0), \ldots, (0, e_{2^n-1})\}$$

is the canonical basis in $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$. By standard abuse of notation, we denote, also $e_0 = (e_0, 0), e_1 = (e_1, 0), \ldots, e_{2^n-1} = (e_{2^n-1}, 0)$,
$e_{2^n} = (0, e_0), \ldots, e_{2^{n+1}-1} = (0, e_{2^n-1})$ in $\mathbb{A}_{n+1}$.

For $\alpha = (a, b) \in \mathbb{A}_n \times \mathbb{A}_n = \mathbb{A}_{n+1}$ we denote $\widetilde{\alpha} = (-b, a)$ (the complexification of α) so $\widetilde{e}_0 = (0, e_0)$ and $\alpha\widetilde{e}_0 = (a, b)(0, e_0) = (-b, a) = \widetilde{\alpha}$. Notice that $\widetilde{\widetilde{\alpha}} = -\alpha$.

The **trace** on $\mathbb{A}_{n+1}$ is the linear map $t_{n+1} : \mathbb{A}_{n+1} \to \mathbb{R}$ given by $t_{n+1}(\alpha) = \alpha + \overline{\alpha} = 2(\text{real part of } \alpha)$ so $t_{n+1}(\alpha) = t_n(a)$ when $\alpha = (a, b) \in \mathbb{A}_n \times \mathbb{A}_n$.

Definition: $\alpha = (a, b)$ in $\mathbb{A}_{n+1}$ <u>is pure</u> if

$$t_{n+1}(\alpha) = t_n(a) = 0.$$

$\alpha = (a, b)$ in $\mathbb{A}_{n+1}$ <u>is doubly pure</u> if it is pure and also $t_n(b) = 0$; i.e., $\widetilde{\alpha}$ is pure in $\mathbb{A}_{n+1}$.

Also $2\langle a, b\rangle = t_n(a\overline{b})$ for $\langle , \rangle$ the inner product in $\mathbb{R}^{2^n}$ (see [Mo$_1$]).

Notice that for a and b pure elements $a \perp b$ if and only if $ab = -ba$.

Notation: $Im(\mathbb{A}_n) = \{e_o\}^{\perp} \subset \mathbb{A}_n$ is the vector subspace consisting of pure elements in $\mathbb{A}_n$; i.e., $Im(\mathbb{A}_n) = \mathrm{Ker}(t_n) = \mathbb{R}^{2^n - 1}$.

$\widetilde{\mathbb{A}}_{n+1} = Im(\mathbb{A}_n) \times Im(\mathbb{A}_n) = \{e_0, \widetilde{e}_0\}^{\perp} = \mathbb{R}^{2^{n+1}-2}$ is the vector subspace consisting of doubly pure elements in $\mathbb{A}_{n+1}$.

Lemma 1.1. For a and b in $\widetilde{\mathbb{A}_n}$ we have that

1) $a\widetilde{e}_0 = \widetilde{a}$ and $\widetilde{e}_0 a = -\widetilde{a}$.

2) $a\widetilde{a} = -||a||^2\widetilde{e}_0$ and $\widetilde{a}a = ||a||^2\widetilde{e}_0$ so $a \perp \widetilde{a}$.

3) $\widetilde{a}b = -\widetilde{ab}$ with a a pure element.

4) $a \perp b$ if and only if $\widetilde{a}b + \widetilde{b}a = 0$.

5) $\widetilde{a} \perp b$ if and only if $ab = \widetilde{b}\widetilde{a}$.

6) $\widetilde{a}b = a\widetilde{b}$ if and only if $a \perp b$ and $\widetilde{a} \perp b$.

Proof: Notice that a is pure if $\overline{a} = -a$ and if $a = (a_1, a_2)$ is doubly pure, then $\overline{a}_1 = -a_1$ and $\overline{a}_2 = -a_2$.

1) $\widetilde{e}_0 a = (0, e_0)(a_1, a_2) = (-\overline{a}_2, \overline{a}_1) = (a_2, -a_1) = -(-a_2, a_1) = -\widetilde{a}$.

2) $a\widetilde{a} = (a_1, a_2)(-a_2, a_1) = (-a_1a_2 + a_1a_2, a_1^2 + a_2^2) = (0, -||a||^2 e_0) = -||a||^2\widetilde{e}_0$.

   Similarly $\widetilde{a}a = (-a_2, a_1)(a_1, a_2) = (-a_2a_1 + a_2a_1, -a_2^2 - a_1^2) = ||a||^2\widetilde{e}_0$.

   Now, since $-2\langle\widetilde{a}, a\rangle = a\widetilde{a} + \widetilde{a}a = 0$ we have $a \perp \widetilde{a}$.

3) $\widetilde{a}b = (-a_2, a_1)(b_1, b_2) = (-a_2b_1 + b_2a_1, -b_2a_2 - a_1b_1)$.

   So $\widetilde{\widetilde{a}b} = (a_1b_1 + b_2a_2, b_2a_1 - a_2b_1) = (a_1, a_2)(b_1, b_2) = ab$ then $-\widetilde{a}b = \widetilde{ab}$.

   Notice that in this proof we only use that $\overline{a}_1 = -a_1$; i.e., a is pure and b doubly pure.

4) $a \perp b \Leftrightarrow ab + ba = 0 \Leftrightarrow ab = -ba \Leftrightarrow \widetilde{ab} = -\widetilde{ba}$.

$\Leftrightarrow -\widetilde{a}b = \widetilde{b}a \Leftrightarrow \widetilde{a}b + \widetilde{b}a = 0$ by (3).

5) $\widetilde{a} \perp b \Leftrightarrow \widetilde{\widetilde{a}}b + \widetilde{b}\widetilde{a} = 0$ (by (4)) $\Leftrightarrow -ab + \widetilde{b}\widetilde{a} = 0$.

6) If $\widetilde{a} \perp b$ and $a \perp b$, then by (3) and (4) $\widetilde{a}b = -\widetilde{ab} = \widetilde{ba} = -\widetilde{b}a = a\widetilde{b}$.

Conversely, put $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ and define $c := (a_1b_1 + b_2a_2)$ and $d := (b_2a_1 - a_2b_1)$ in $\mathbb{A}_{n-1}$.

Then $a\widetilde{b} = (a_1, a_2)(-b_2, b_1) = (-a_1b_2 + b_1a_2, b_1a_1 + a_2b_2)$ so $a\widetilde{b} = (-\overline{d}, \overline{c})$.

Now $ab = (a_1, a_2)(b_1, b_2) = (a_1b_1 + b_2a_2, b_2a_1 - a_2b_1) = (c, d)$, so $\widetilde{ab} = (-d, c)$ and then $\widetilde{a}b = (d, -c)$.

Thus, if $a\widetilde{b} = \widetilde{a}b$ then $\overline{c} = -c$ and $d = -\overline{d}$. Then

$$\begin{aligned} t_n(ab) &= t_{n-1}(c) = c + \overline{c} = 0 \text{ and } a \perp b \\ t_n(\widetilde{a}b) &= t_{n-1}(d) = d + \overline{d} = 0 \text{ and } \widetilde{a} \perp b. \end{aligned}$$

Q.E.D

Corollary 1.2 For each $a \neq 0$ in $\widetilde{\mathbb{A}}_n$. The four dimensional vector subspace generated by $\{e_0, \widetilde{a}, a, \widetilde{e}_0\}$ is a copy of $\mathbb{A}_2 = \mathbb{H}$.(We denote it by $\mathbb{H}_a$).

Proof: We suppose that $||a|| = 1$, otherwise we take $\frac{a}{||a||}$. Construct the following multiplication table.

	e_0	$\widetilde{a}$	a	$\widetilde{e}_0$
e_0	e_0	$\widetilde{a}$	a	$\widetilde{e}_0$
$\widetilde{a}$	$\widetilde{a}$	$-e_0$	$+\widetilde{e}_0$	$-a$
a	a	$-\widetilde{e}_0$	$-e_0$	$\widetilde{a}$
$\widetilde{e}_0$	a	$\widetilde{e}_0$	$-\widetilde{a}$	$-e_0$

By lemma 2.1. $a\widetilde{e}_0 = \widetilde{a}$; $\widetilde{e}_0a = -\widetilde{a}$; $\widetilde{a}\widetilde{e}_0 = \widetilde{\widetilde{a}} = -a$; $\widetilde{e}_0\widetilde{a} = -\widetilde{\widetilde{a}} = a$; $a\widetilde{a} = -\widetilde{e}_0$ and $\widetilde{a}a = \widetilde{e}_0$.

But this is the multiplication table of $\mathbb{A}_2 = \mathbb{H}$ identifying $e_0 \leftrightarrow 1$, $\widetilde{a} \leftrightarrow \hat{i}$, $a \leftrightarrow \hat{j}$ and $\widetilde{e}_0 \leftrightarrow \hat{k}$.

Q.E.D.

§ **2. Monomorphism from $\mathbb{A}_m$ to $\mathbb{A}_n$.**

Throughout this chapter $1 \leq m \leq n$.

Definition. An algebra monomorphism from $\mathbb{A}_m$ to $\mathbb{A}_n$ is a linear monomorphism $\varphi : \mathbb{A}_m \to \mathbb{A}_n$ such that

i) $\varphi(e_0) = e_0$ (the first e_0 is in $\mathbb{A}_m$ and the second e_0 in $\mathbb{A}_n$)

ii) $\varphi(xy) = \varphi(x)\varphi(y)$ for all x and y in $\mathbb{A}_m$.

By definition we have that $\varphi(re_0) = r\varphi(e_0)$ for all r in $\mathbb{R}$ so $\varphi(Im(\mathbb{A}_m)) \subset \varphi(Im(\mathbb{A}_n))$ and $\varphi(\overline{x}) = \overline{\varphi(x)}$.

Therefore $||\varphi(x)||^2 = \varphi(x)\overline{\varphi(x)} = \varphi(x)\varphi(\overline{x}) = \varphi(x\overline{x}) = \varphi(||x||^2) = ||x||^2$ for all $x \in \mathbb{A}_m$ and $||\varphi(x)|| = ||x||$ and φ is an orthogonal linear transformation from $\mathbb{R}^{2^m-1}$ to $\mathbb{R}^{2^n-1}$.

The trivial monomorphism is the one given by $\varphi(x) = (x, 0, 0, \ldots, 0)$ for $x \in \mathbb{A}_m$ and 0 in $\mathbb{A}_m$ $\quad (2^{n-m-1}$-times).

$\mathcal{M}(\mathbb{A}_m, \mathbb{A}_n)$ denotes the set of algebra monomorphisms from $\mathbb{A}_m$ to $\mathbb{A}_n$.

For $m = n$, $\mathcal{M}(\mathbb{A}_m; \mathbb{A}_n) = \mathrm{Aut}(\mathbb{A}_n)$ the group of algebra automorphisms of $\mathbb{A}_n$

Proposition 2.1. $\mathcal{M}(\mathbb{A}_1; \mathbb{A}_n) = S(Im(\mathbb{A}_n)) = S^{2^n-2}$.

Proof: $\mathbb{A}_1 = \mathbb{C} = Span\{e_0, e_1\}$.

If $x \in \mathbb{A}_1$ then $x = re_0 + se_1$ and for $w \in Im(\mathbb{A}_n)$ with $||w|| = 1$ we have that $\varphi_w(x) = re_0 + sw$ define an algebra monomorphism from $\mathbb{A}_1$ to $\mathbb{A}_n$. This can be seen by direct calculations, recalling that, Center $(\mathbb{A}_n) = \mathbb{R}$ for all n and that every associator with one real entrie vanish.

$$\begin{aligned}
\varphi_w(x)\varphi_w(y) &= (re_0 + sw)(pe_0 + qw) = (rp + sqw^2)e_0 + (rq + sp)w \\
&= (rp - sq)e_0 + (rq + sp)w \\
&= \varphi_w(x)\varphi_w(y)
\end{aligned}$$

when $y = pe_0 + qe_1$ and p and q in $\mathbb{R}$. Clearly $\varphi_w(e_0) = e_0$.

Conversely, for $\varphi \in \mathcal{M}(\mathbb{A}_1; \mathbb{A}_n)$, set $w = \varphi(e_1)$ in $\mathbb{A}_n$ so $||w|| = 1$ and $\varphi_w = \varphi$.

Q.E.D.

Remark: In particular, we have that

$$Aut(\mathbb{A}_1) = S^0 = \mathbb{Z}/2 = \{\text{Identity, conjugation}\} = \{\varphi_{e_1}, \varphi_{-e_1}\}.$$

To calculate $\mathcal{M}(\mathbb{A}_2;\mathbb{A}_n)$ for $n \geq 2$ we need to recall (see [Mo$_2$]).

Definition: For a and b in $\mathbb{A}_n$. We said that $\underline{a \text{ alternate with } b}$, we denote it by $a \rightsquigarrow b$, if $(a,a,b)=0$.

We said that $\underline{a \text{ alternate strongly with } b}$, we denote it by $a \leftrightsquigarrow b$, if $(a,a,b)=0$ and $(a,b,b)=0$.

Clearly a alternate strongly with e_0 for all a in $\mathbb{A}_n$ and if a and b are linearly dependent then $a \leftrightsquigarrow b$ (by flexibility).

Also,by definition,a is an alternative element if and only if $a \rightsquigarrow x$ for all x in $\mathbb{A}_n$.

By Lemma 1.1 (1) and (2) we have that for any doubly pure element a in $\mathbb{A}_n$ $(a,a,\tilde{e}_0)=0$ and (by the above remarks) $\tilde{e}_0$ alternate strongly with any a in $\mathbb{A}_n$.

For a and b pure elements in $\mathbb{A}_n$, we define the vector subspace of $\mathbb{A}_n$

$$V(a;b) = \mathrm{Span}\{e_0, a, b, ab\}.$$

Also we identify the Stiefel manifold $V_{2^n-1,2}$ as

$$\{(a,b) \in Im(\mathbb{A}_n) \times Im(\mathbb{A}_n) | a \perp b, ||a|| = ||b|| = 1\}$$

Lemma 2.2. If $(a,b) \in V_{2^n-1,2}$ and $a \leftrightsquigarrow b$ then $V(a;b) = \mathbb{A}_2 = \mathbb{H}$ the quaternions.

Proof: Suppose that $(a,b) \in V_{2^n-1,2}$ and that $(a,a,b)=0$ then we have

$$\begin{aligned}
\langle ab, a\rangle &= \langle b, \overline{a}a\rangle = \langle b, ||a||^2 e_0\rangle = ||a||^2\langle b, e_0\rangle = 0 \\
\langle ab, a\rangle &= \langle a, b\overline{b}\rangle = \langle a, ||b||^2 e_0\rangle = ||b||^2\langle a, e_0\rangle = 0 \\
||ab||^2 = \langle ab, ab\rangle &= \langle \overline{a}(ab), b\rangle = \langle -a(ab), b\rangle = \langle -a^2 b, b\rangle \\
&= -a^2\langle b, b\rangle = ||a||^2||b||^2 = 1
\end{aligned}$$

so $\{e_0, a, b, ab\}$ is an orthonormal set of vectors in $\mathbb{A}_n$.

Finally using also that $(a,b,b)=0$ and $ab=-ba$ we may check by direct calculations that the multiplication table of $\{e_0,a,b,ab\}$ coincides with the one of the quaternions and by the identification $e_0 \mapsto e_0, a \mapsto e_1, b \mapsto e_2$ and $ab \mapsto e_3$ we have an algebra isomorphism between $\mathbb{A}_2 = \mathbb{H}$ and $V(a;b)$.

Q.E. D.

Proposition 2.3. $\mathcal{M}(\mathbb{A}_2;\mathbb{A}_n) = \{(a,b) \in V_{2^n-1,2} | a \leftrightsquigarrow b\}$ for $n \geq 2$.

In particular

$$Aut(\mathbb{A}_2) = \mathcal{M}(\mathbb{A}_2; \mathbb{A}_2) = V_{3,2} = SO(3)$$

and

$$\mathcal{M}(\mathbb{A}_2, \mathbb{A}_3) = V_{7,2}.$$

Proof. The inclusion "$\supset$" follows from Lemma 2.2. Conversely suppose that $\varphi \in \mathcal{M}(\mathbb{A}_2, \mathbb{A}_n)$ then $\varphi(e_0) = e_0, (\varphi(e_1), \varphi(e_2)) \in V_{2^n-1,2}$ and

$$V(\varphi(e_1), \varphi(e_2)) = \mathrm{Im}\varphi = \mathbb{H} \subset \mathbb{A}_n.$$

Since $\mathbb{A}_2$ is an associative algebra and $\mathbb{A}_3$ is an alternative algebra we have that $a \leftrightsquigarrow b$ for any two elements in $\mathbb{A}_n$ for $n = 2$ or $n = 3$.

Q.E.D.

Remark. Recall that $\tilde{\mathbb{A}}_n = \{e_0, \tilde{e}_0\}^\perp = \mathbb{R}^{2^n-2}$ denotes the vector subspace of doubly pure elements. Since $a \leftrightsquigarrow \tilde{e}_0$ for any element in $\tilde{\mathbb{A}}_n$, we have that, if $a \in S(\tilde{\mathbb{A}}_n)$ i.e., $||a|| = 1$ then $(a, \tilde{e}_0) \in V_{2^n-1,2}$ and the assignment $a \mapsto (a, \tilde{e}_0)$ define an inclusion from $S(\tilde{\mathbb{A}}_n) = S^{2^n-3} \hookrightarrow \mathcal{M}(\mathbb{A}_2; \mathbb{A}_n) \subset V_{2^n-1,2}$ which resembles "the bottom cell" inclusion in $V_{2^n-1,2}$.

To deal with the cases $3 = m \leq n$ we have to use the notion of a <u>special triple</u> (see [Wh] and [Mo$_1$]).

Definition: A set $\{a, b, c\}$ in $Im(\mathbb{A}_n)$ is <u>a special triple</u> if

(i) $\{a, b, c\}$ is an orthonormal set

(ii) $a \leftrightsquigarrow b, a \leftrightsquigarrow c$ and $b \leftrightsquigarrow c$ i.e. its elements alternate strongly, pairwise.

(iii) $c \in V(a; b)^\perp \subset \mathbb{A}_n$.

Now is easy to see that if $\{a, b, c\}$ is a special triple then $V(a; b); V(a, c); V(b, c)$ are isomorphic to $\mathbb{A}_2$.

For a special triple $\{a, b, c\}$ consider the following vector subspace of $\mathbb{A}_n$

$$\o(a; b; c) := Span\{e_0, a, b, ab, c(ab), cb, ac, c\}.$$

Proposition 2.4: For a special triple $\{a, b, c\}$ in $\mathbb{A}_n$ and $n \geq 3$; $\o(a, b, c)$ is an eight-dimensional vector subspace isomorphic, as algebra, to $\mathbb{A}_3 = \o$ the octonions and $\mathcal{M}(\mathbb{A}_3, \mathbb{A}_n) = \{(a, b, c) \in (\mathbb{A}_n)^3 | \{a, b, c\} \text{ special triple}\}$.

Proof: We know that all elements in $\{e_0, a, b, ab, c(ab), cb, ac, c\}$ are of norm one and also

$$\langle c(ab), a \rangle = -\langle ab, ac \rangle = \langle a(ab), c \rangle = \langle a^2 b, c \rangle = a^2 \langle b, c \rangle = 0.$$

Similarly $(c(ab)) \perp b$ and $(c(ab)) \perp c$. Thus $\{e_0, a, b, ab, c(ab), cb, ac, c\}$ is an orthonormal set of vectors and $\o(a;b;c)$ is eight-dimensional. To see that $\o(a;b;c) \cong \mathbb{A}_3$ we have to construct the corresponding multiplication table,which is, a routine calculation.(See [Mo$_2$])

Conversely, if $\varphi \in \mathcal{M}(\mathbb{A}_3, \mathbb{A}_n)$ then $a = \varphi(e_1), b = \varphi(e_2)$ and $c = \varphi(e_7)$ form an special triple, when $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is the canonical basis in $\mathbb{A}_3$, and we recall that $e_1 e_2 = e_3, e_7 e_3 = e_4, e_7 e_2 = e_5$ and $e_1 e_7 = e_6$ in $\mathbb{A}_3$. Q.E.D.

Remark. For $n = 3, \mathbb{A}_3$ is an alternative algebra so a special triple in $\mathbb{A}_3$ is every triple such that

(i) $\{a, b, c\}$ is orthonormal

(ii) $c \perp (ab)$.

So Proposition 4.4 gives the construction of $G_2 = Aut(\mathbb{A}_3)$ as in [Wh] and the assignment

$$\begin{array}{rcl} G_2 = Aut(\mathbb{A}_3) & \xrightarrow{\pi} & \mathcal{M}(\mathbb{A}_2, \mathbb{A}_3) = V_{7,2} \\ (a, b, c) & \mapsto & (a, b) \end{array}$$

is the known fibration $G_2 \xrightarrow{\pi} V_{7,2}$ with fiber S^3.

Remark. Suppose that $n \geq 4$ and that $\{a, b, c\}$ is a special triple in $\mathbb{A}_n$ so $\o(a;b;c)$ is the image of some algebra monomorphism from $\mathbb{A}_3$ to $\mathbb{A}_n$ and any orthonormal triple $\{x, y, z\}$ of pure elements in $\o(a;b;c)$ with $z \perp (xy)$ is also a special triple in $\mathbb{A}_n$ and

$$\o(x;y;z) = \o(a;b;c).$$

Definition: For $1 \leq m \leq n$. $\varphi \in \mathcal{M}(\mathbb{A}_m, \mathbb{A}_n)$ $\underline{\text{is a type I monomorphism}}$ if $\widetilde{e}_0 \in$ (Image of φ) $\subset \mathbb{A}A_n$.

Since $\tilde{e}_0 = e_{2^{n-1}}$ in $\mathbb{A}A_n$ then the trivial monomorphism is $\underline{\text{not}}$ a type I monomorphism unless $n = m$, because by definition its image is generated by $\{e_0, e_1, \ldots, e_{2^{m-1}-1}\}$.

Denote by $\mathcal{M}_1(\mathbb{A}A_m; \mathbb{A}A_n) = \{\varphi \in \mathcal{M}(\mathbb{A}A_m; \mathbb{A}A_n) | \tilde{e}_0 \in (\mathrm{Im}\varphi)\}$ the subset of all type I monomorphisms, clearly $\mathcal{M}_1(\mathbb{A}A_n, \mathbb{A}A_n) = \mathcal{M}(\mathbb{A}A_n; \mathbb{A}A_n) = \mathrm{Aut}(\mathbb{A}A_n)$.

Using proposition 2.1 we may verify that for $n \geq 2$.

$$\mathcal{M}_1(\mathbb{A}A_1, \mathbb{A}A_n) = \{re_0 + s\tilde{e}_0 | r^2 + s^2 = 1\} = S^1.$$

Also,by Lemma 2.2, for a non-zero $a \in \tilde{\mathbb{A}A_n}$ we have that $V(a;\tilde{e}_0) = \mathbb{A}H_a$ and

$$\mathcal{M}_1(\mathbb{A}A_2;\mathbb{A}A_n) = S(\tilde{\mathbb{A}A_n}) = S^{2^n-3}$$

. In particular

$$S^5 = \mathcal{M}_1(\mathbb{A}A_2;\mathbb{A}A_3) \subset \mathcal{M}(\mathbb{A}A_2;\mathbb{A}A_3) = V_{7,2}$$

is "the bottom cell" of $V_{7,2}$.

Also we can check,using the fact that $a \rightsquigarrow \tilde{e}_0$ for all $a \in \tilde{\mathbb{A}A_n}$, that

$$\mathcal{M}_1(\mathbb{A}A_3;\mathbb{A}A_n) = \mathcal{M}(\mathbb{A}A_2;\mathbb{A}A_n) \cap V_{2^n-2,2}$$

where $V_{2^n-2,2} = \{(a,b) \in V_{2^n-1,2} | a \text{ and } b \text{ are in } \tilde{\mathbb{A}A_n}\}$.

Definition: A type I monomorphism $\varphi \in \mathcal{M}_1(\mathbb{A}A_m;\mathbb{A}A_{n+1})$ is of type II if

$$e_{2^{n-1}} := \varepsilon \in (\text{Im} \quad \varphi) \subset \mathbb{A}A_{n+1}.$$

By definition if $\varphi \in \mathcal{M}_1(\mathbb{A}A_m;\mathbb{A}A_{n+1})$ then $\tilde{e}_0 \in (Im\ \varphi) \subset \mathbb{A}A_{n+1}$ and if we also assume that $\varepsilon \in (Im\varphi) \subset \mathbb{A}A_{n+1}$ then $\varepsilon\tilde{e}_0 = \tilde{\varepsilon}$ and $\mathbb{A}H_\varepsilon := Span\{e_0, \tilde{\varepsilon}, \varepsilon, \tilde{e}_0\}$ lies in $(Im\,\varphi) \subset \mathbb{A}A_{n+1}$; therefore

$$\varphi \in \mathcal{M}(\mathbb{A}A_m, \mathbb{A}A_{n+1}) \text{ is of type II if and only if } \mathbb{A}H_\varepsilon \subset (Im\ \varphi).$$

Denote $\mathcal{M}_2(\mathbb{A}A_m, \mathbb{A}A_{n+1}) = \{\varphi \in \mathcal{M}(\mathbb{A}A_m;\mathbb{A}A_{n+1}) | \varphi \text{ is type II}\}$.

Theorem 2.5 For $n \geq 3$.

$$\mathcal{M}_2(\mathbb{A}A_3;\mathbb{A}A_{n+1}) = \mathbb{A}CP^{2^{n-1}-1} \cup \overline{X}_n$$

where $\mathbb{A}CP^m$ is the complex projective space in $\mathbb{A}C^m$ and

$$\overline{X}_n = \{(x,y) \in \mathbb{A}A_n \times \mathbb{A}A_n | xy = 0, x \neq 0 \text{ and } y \neq 0\}.$$

In particular for $n = 3$ $\quad \overline{X}_3 = \Phi$ (empty set) and $\mathcal{M}_2(\mathbb{A}A_3;\mathbb{A}A_4) = \mathbb{A}CP^3$.

Proof: Suppose that $\varphi : \mathbb{A}A_3 \to \mathbb{A}A_{n+1}$ is an algebra monomorphism for $n \geq 3$ with $\mathbb{A}H_\varepsilon \subset (Im\,\varphi)$. So $Im\,\varphi$ is isomorphic to $\mathbb{A}O = \mathbb{A}A_3$, as algebras, then there is a non-zero $\alpha \in \mathbb{A}H_\varepsilon^\perp \subset \tilde{A}_{n+1}$ such that

$$Im\,\varphi = \mathbb{A}O_\alpha := Span\{e_0, \tilde{\varepsilon}, \varepsilon, \tilde{e}_0, \tilde{\alpha}, \alpha\varepsilon, \tilde{\varepsilon}\alpha, \alpha\} \subset \mathbb{A}A_{n+1}$$

Suppose that $\alpha = (a, b) \in \tilde{\mathbb{A}A_n} \times \tilde{\mathbb{A}A_n} = \mathbb{A}H_\varepsilon^\perp \subset \mathbb{A}A_{n+1}$ and that $a \neq 0$ (similarly we may assume $b \neq 0$).

Now $Im\,\varphi = \mathbb{A}O_\alpha = \mathbb{A}A_3$ if and only if

$$(\alpha, \alpha, \varepsilon) := \alpha^2\varepsilon - \alpha(\alpha\varepsilon) = 0 \quad \text{i.e.} \quad -||\alpha||^2\varepsilon = \alpha(\alpha\varepsilon).$$

Using Lemma 1.1 we have that

$$\begin{aligned}
\alpha(\alpha\varepsilon) &= (a, b)[(a, b)(\tilde{e}_0, 0)] = (a, b)[(\tilde{a}, -\tilde{b})] = \\
&= (a\tilde{a} - \tilde{b}b, -\tilde{b}a - b\tilde{a}) = (-||a||^2\tilde{e}_0 - ||b||^2\tilde{e}_0, 0) - (0, (b\tilde{e}_0)a - b(\tilde{e}_0 a)) \\
&= -||\alpha||^2\varepsilon + (0, (a, \tilde{e}_0, b)).
\end{aligned}$$

Therefore $(a, \tilde{e}_0, b) = 0$ in $\mathbb{A}A_n$ if and only if $(\alpha, \alpha, \varepsilon) = 0$ in $\mathbb{A}A_{n+1}$.

Since $\mathbb{A}A_n = \mathbb{A}H_a \oplus \mathbb{A}H_a^\perp$ we have that $b = c + d$ where $c \in \mathbb{A}H_a$ and $d \in \mathbb{A}H_a^\perp$ with c doubly pure i.e., $c \in Span\{a, \tilde{a}\} \subset \mathbb{A}H_a$.

Since $\mathbb{A}H_a$ is associative we have that

$$\begin{aligned}
(a, \tilde{e}_0, b) &= (a, \tilde{e}_0, c + d) = (a, \tilde{e}_0, c) + (a, \tilde{e}_0, d) = 0 + (a, \tilde{e}_0, d) \\
&= (a, \tilde{e}_0, d) = \tilde{a}d + \tilde{ad} = 2\tilde{a}d
\end{aligned}$$

by Lemma 1.1 (1), (2) and (6).

Therefore $(\alpha, \alpha, \varepsilon) = 0$ in $\mathbb{A}A_{n+1}$ if and only if $ad = \tilde{a}d = 0$ in $\mathbb{A}A_n$.

So, we have two cases: namely $d = 0$ and $d \neq 0$ in $\mathbb{A}A_n$.

Suppose $d = 0$. Thus $b = c \in \mathbb{A}H_a$ and $b \in Span\{a, \tilde{a}\}$ so (a, b) determine a complex line in $\tilde{\mathbb{A}A_n} = \mathbb{R}^{2^n-2} = \mathbb{A}C^{2^{n-1}-1}$ and $\alpha \in \mathbb{A}CP^{2^{n-1}-1}$.

Suppose $d \neq 0$ (so $b \neq 0$ and $a \neq 0$).

Thus $ad = 0$ and $(a, d) \in \overline{X}_n$.

Q.E.D.

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